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Harmonic oscillator with variable mass

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Abstract. A general treatment of the quantal harmonic oscillator with variable mass is given. Various examples additional to those obtained by Colegrave and Abdalla for which a closed form solution is possible are given.

1. Introduction

In recent papers of Colegrave and Abdalla (1981a,b, 1982) particular examples of an harmonic oscillator with variable mass parameter have been treated. The two examples were an exponentially changing mass (Colegrave and Abdalla 1981b) and a strongly pulsating mass (Colegrave and Abdalla 1982). As discussed by Colegrave and Abdalla (1981a) the problem arises in a Fabray–Pérot cavity.

In this note we discuss an alternative treatment of the general problem of the variable mass oscillator with Hamiltonian

$$H(t)=\tfrac{1}{2}p^2/M(t)+\tfrac{1}{2}M(t)\omega^2q^2 \tag{1.1}$$

where $M(t)$ is the variable mass and ω, q and p have the usual meanings. We believe that this treatment has some advantage over that used in the work of Colegrave and Abdalla, in that it is more direct and leads directly to the solution of the Schrödinger equation corresponding to the Hamiltonian (1.1) which is

$$\{-[\hbar^2/2M(t)]\,\partial^2/\partial q^2+\tfrac{1}{2}M(t)\omega^2q^2\}\psi=\mathrm{i}\hbar\,\partial\psi/\partial t. \tag{1.2}$$

There are two points at which the treatment contained herein differs from that found in Colegrave and Abdalla (1982). In contrast to their approach, firstly a change of time scale is introduced and then a generalised canonical transformation (Lewis and Leach 1982, Munier *et al* 1981) is made. This reduces the Hamiltonian to that of a time-independent harmonic oscillator for which the solution of the corresponding Schrödinger equation is well known. The solution of the original Schrödinger equation (1.2) follows automatically using the integral transform method of Wolf (1979). In this particular instance, because the space part of the generalised canonical transformation is a point transformation, the integral transform reduces to a geometric transform (Leach 1977) so that a possibly awkward integration is avoided.

The Hamiltonian (1.1), which is the basis for the treatment outlined in this note, is particularly simple in structure and accords some simplifications in the calculations. The identification of the required form of the generalised canonical transformation is straightforward since it is already known in the literature (cf Leach 1977). The collapse of the integral transform to a geometric transform follows from the pointlike nature of that transformation. However, the method outlined here may be extended to Hamiltonians more complex than (1.1). A positive definite quadratic Hamiltonian (with additional linear terms if so desired) will require an integral transform in general. If the Hamiltonian is that of a nonlinear system of specific time-dependent form (cf Leach 1981), the same treatment as given here is applicable. For such a system the difficulty lies in the solution of a time-independent Schrödinger equation with a general potential. Unfortunately closed form solutions to such Schrödinger equations are not usually available.

The purpose of this note is to demonstrate the solution of the Schrödinger equation (1.2) by a particular direct method. This is done for a general time-varying mass function $M(t)$.

2. Canonical transformation of the Hamiltonian

A change of time scale for a Hamiltonian system may be regarded as a particular type of generalised canonical transformation. For the Hamiltonian (1.1) it is

$$(q, p, t) \to \left(q', p', t' : q' = q, p' = p, t' = \int^{t} M^{-1}(s)\,\mathrm{d}s\right). \tag{2.1}$$

The presence of simple or multiple zeros of $M(t)$ for finite t merely means that the time scale in the primed space has been expanded by comparison with that of the original space. An example of this is given in the discussion of § 7.

The physical system described by (1.1) is now equivalently described by

$$H'(t') = \tfrac{1}{2}p'^2 + \tfrac{1}{2}N^2(t')\omega^2 q'^2 \tag{2.2}$$

where

$$N[t'(t)] = M(t). \tag{2.3}$$

We recognise that the new Hamiltonian (2.2) is just that of the time-dependent harmonic oscillator about which there exists an extensive literature. The Hamiltonian (2.2) may be transformed into that of the time-independent harmonic oscillator

$$\bar{H}(T) = \tfrac{1}{2}P^2 + \tfrac{1}{2}Q^2 \tag{2.4}$$

by means of a generalised canonical transformation which is a point transformation in the space variables. Rather than just quote the transformation, it may be of interest to some readers to view its derivation. The derivation is in two parts.

A linear point canonical transformation from (q', p') to (Q', P') has the form

$$Q' = q'/\rho, \qquad P' = \rho p' - \sigma q', \tag{2.5}$$

where ρ and σ are as yet unspecified functions of t'. The type-zero generating function (cf Lewis and Leach 1982) is determined according to

$$\partial F_0/\partial q' = p' - P'\,\partial Q'/\partial q, \qquad \partial F_0/\partial p' = -P'\,\partial Q'/\partial p', \tag{2.6}$$

and, after the expressions for Q' and P' in (2.5) are substituted into (2.6), we have

$$F_0(q', p', t') = \tfrac{1}{2}(\sigma/\rho)q'^2 + \eta(t'). \tag{2.7}$$

The additive function of t' is not required and is hereafter abandoned. The transformed

Hamiltonian is determined by

$$\bar{H}'(t') = H'(t') + P'\,\partial Q'/\partial t' + \partial F/\partial t' \tag{2.8}$$

and, after the appropriate substitutions and rearrangements are made, is

$$\bar{H}'(t') = \tfrac{1}{2}\rho^{-2}P'^2 + (\sigma/\rho - \dot{\rho}/\rho)P'Q' + \tfrac{1}{2}(N^2\omega^2\rho^2 + \sigma^2 + \dot{\sigma}\rho - \sigma\dot{\rho})Q'^2 \tag{2.9}$$

where a dot indicates differentiation with respect to t'. The term in $P'Q'$ may be removed by the identification of σ with $\dot{\rho}$ and (2.9) becomes

$$\bar{H}'(t') = \tfrac{1}{2}\rho^{-2}P'^2 + \tfrac{1}{2}(N^2\omega^2\rho^2 + \ddot{\rho}\rho)Q'^2. \tag{2.10}$$

We now require $\rho(t')$ to be a solution of the so-called auxiliary equation (Eliezer and Gray 1976)

$$\ddot{\rho} + N^2\omega^2\rho = 1/\rho^3. \tag{2.11}$$

Then (2.10) reads

$$\bar{H}'(t') = \tfrac{1}{2}\rho^{-2}(P'^2 + Q'^2). \tag{2.12}$$

The generalised canonical transformation

$$(Q', P', t') \to \left(Q, P, T\colon Q = Q', P = P', T = \int^{t'} \rho^{-2}(s)\,\mathrm{d}s\right) \tag{2.13}$$

yields the time-independent form (2.4).

In summary, the transformation from (2.2) to (2.4) is achieved by means of the generalised canonical transformation

$$(q', p', t') \to \left(Q, P, T\colon Q = q'/\rho, P = \rho p' - \dot{\rho}q', T = \int^{t'} \rho^{-2}(s)\,\mathrm{d}s\right) \tag{2.14}$$

where $\rho(t')$ is a solution of (2.11).

Remark. In the case of a general quadratic Hamiltonian

$$H(t) = a(t)p^2 + 2b(t)qp + c(t)q^2 \tag{2.15}$$

the procedure is almost precisely the same as that given above. Firstly the $a(t)$ is eliminated by means of a change of time variable and then a linear transformation of the form

$$(q, p) \to (Q, P\colon Q = \alpha q + \beta p, P = \gamma q + \delta p, \alpha\delta - \beta\gamma = 1) \tag{2.16}$$

is employed. The cross term in Q, P is eliminated by putting its coefficient equal to zero and an auxiliary equation obtained just as (2.11) was. It should be emphasised that for Schrödinger equation treatments, the signatures of the original and transformed Hamiltonians must be the same.

3. Solution of the Schrödinger equation

Corresponding to the Hamiltonians (1.1), (2.2), (2.12) and (2.4) we have respectively the following Schrödinger equations:

$$\{-[\hbar^2/2M(t)]\,\partial^2/\partial q^2 + \tfrac{1}{2}M(t)\omega^2 q^2\}\psi = \mathrm{i}\hbar\,\partial\psi/\partial t, \tag{3.1}$$

$$(-\tfrac{1}{2}\hbar'^2\,\partial^2/\partial q'^2+\tfrac{1}{2}N^2(t')\omega^2 q'^2)\psi'=\mathrm{i}\hbar\,\partial\psi'/\partial t', \tag{3.2}$$

$$\rho^{-2}(-\tfrac{1}{2}\hbar^2\,\partial^2/\partial Q'^2+\tfrac{1}{2}Q'^2)\bar{\psi}'=\mathrm{i}\hbar\,\partial\bar{\psi}'/\partial t', \tag{3.3}$$

$$(-\tfrac{1}{2}\hbar^2\,\partial^2/\partial Q^2+\tfrac{1}{2}Q^2)\bar{\psi}=\mathrm{i}\hbar\,\partial\bar{\psi}/\partial T. \tag{3.4}$$

The eigenfunctions of (3.4) are well known. We have

$$\bar{\psi}_n(Q,T)=[2^n n!(\pi\hbar)^{1/2}]^{-1/2}\exp[-\mathrm{i}(n+\tfrac{1}{2})T]\exp[-\tfrac{1}{2}Q^2/\hbar]H_n[Q/\hbar^{1/2}] \tag{3.5}$$

where $H_n(\cdot)$ is the nth Hermite polynomial. The solution of (3.3) is related to that of (3.4) according to

$$\bar{\psi}'_n(Q',t')=\bar{\psi}_n[Q',T(t')]. \tag{3.6}$$

The solution of (3.2) is obtained from that of (3.3) by means of the integral transform method of Wolf (1979). Since the canonical transformation from the Hamiltonian (2.2) to (2.12) is a point transformation, the integral transform collapses to a geometric transform and we have (cf Leach 1977)

$$\psi'_n(q',t')=|\rho(t')|^{-1/2}\exp[\tfrac{1}{2}\mathrm{i}\rho^{-1}(t')\dot{\rho}(t')q'^2/\hbar]\bar{\psi}_n[\rho^{-1}(t')q',T(t')]. \tag{3.7}$$

Finally the solution of (3.1) follows from that of (3.2) by a relationship of the same form as that given in (3.6). We have

$$\psi_n(q,t)=|\rho[t'(t)]|^{-1/2}\exp\{\tfrac{1}{2}\mathrm{i}\rho^{-1}[t'(t)]\dot{\rho}[t'(t)]q^2/\hbar\}\bar{\psi}_n[\rho^{-1}[t'(t)]q,T[t'(t)]] \tag{3.8}$$

where it should be remembered that $\dot{\rho}$ is the derivative of ρ with respect to t'. If we define

$$\nu(t)=\rho[t'(t)],\qquad \tau(t)=T[t'(t)], \tag{3.9}$$

(3.8) becomes

$$\psi_n(q,t)=|\nu|^{-1/2}\exp(\tfrac{1}{2}\mathrm{i}\nu^{-1}M\dot{\nu}q^2/t)\bar{\psi}_n[\nu^{-1}q,\tau], \tag{3.10}$$

where $\dot{\nu}$ indicates the derivative of $\nu(t)$ with respect to t.

It is interesting to note that the structure of the right-hand side of (3.10) is not really much more complicated than that of a time-independent harmonic oscillator.

4. Expectation values and matrix elements

Although the Schrödinger wavefunction (3.10) is not much more complicated in form than that for the time-independent oscillator, the extra exponential and the more involved nature of the arguments in the various functions do add to the algebraic difficulty of computing matrix elements and expectation values. The computational task may be reduced by using the canonical transformations relating the classical variables (q,p) and (Q,P). If we write the states ψ_n (3.10) and $\bar{\psi}_n$ (3.5) in Dirac notation as $|n\rangle$ and $|\bar{n}\rangle$ respectively we have the simple result that

$$\langle n|f(q,p)|n\rangle=\langle\bar{n}|F(Q,P)|\bar{n}\rangle \tag{4.1}$$

where

$$F[Q(q,t),P(q,p,t)]=f(q,p). \tag{4.2}$$

Naturally attention must be paid to ordering. In particular we have the well known formulae for expectation values:

$$\langle\bar n|Q|\bar n\rangle=0,\qquad \langle\bar n|P|\bar n\rangle=0,\qquad \langle\bar n|Q^2|\bar n\rangle=\hbar(n+\tfrac12),$$
$$\langle\bar n|PQ+QP|\bar n\rangle=0,\qquad \langle\bar n|P^2|\bar n\rangle=\hbar(n+\tfrac12). \tag{4.3}$$

It is then a trivial exercise to calculate that

$$\begin{aligned}
&\langle n|q|n\rangle=\langle\bar n|\nu Q|\bar n\rangle=0,\qquad \langle n|p|n\rangle=\langle\bar n|P/\nu+M\dot\nu Q|\bar n\rangle=0,\\
&\langle n|q^2|n\rangle=\langle\bar n|\nu^2Q^2|\bar n\rangle=\nu^2\hbar(n+\tfrac12),\\
&\langle n|qp+pq|n\rangle=\langle\bar n|\nu Q(P/\nu+M\dot\nu Q)+(P/\nu+M\dot\nu Q)\nu Q|\bar n\rangle\\
&\qquad=2M\nu\dot\nu\langle\bar n|Q^2|\bar n\rangle+\langle\bar n|QP+PQ|\bar n\rangle=2M\nu\dot\nu\hbar(n+\tfrac12),\\
&\langle n|p^2|n\rangle=\langle\bar n|P^2/\nu^2+(M\dot\nu/\nu)(PQ+QP)+M^2\dot\nu^2Q^2|\bar n\rangle=(\nu^{-2}+M^2\dot\nu^2)\hbar(n+\tfrac12).
\end{aligned} \tag{4.4}$$

We note that the Heisenberg uncertainty relation is

$$\langle(\Delta p)^2\rangle\langle(\Delta q)^2\rangle=(1+M^2\nu^2\dot\nu^2)\hbar^2(n+\tfrac12)^2 \tag{4.5}$$

which is greater than that for the corresponding time-independent problem.

For completeness we write down the matrix elements. For the time-independent oscillator we have

$$\begin{aligned}
&\langle\bar n|Q^2|\bar n'\rangle=\hbar\{\tfrac12[n'(n'-1)]^{1/2}\delta_{n,n'-2}+(n+\tfrac12)\delta_{n,n'}+\tfrac12[n(n-1)]^{1/2}\delta_{n,n'+2}\},\\
&\langle\bar n|QP+PQ|\bar n'\rangle=\mathrm i\hbar\{[n(n-1)]^{1/2}\delta_{n,n'+2}-[n'(n'-1)]^{1/2}\delta_{n,n'-2}\},\\
&\langle\bar n|P^2|\bar n'\rangle=\hbar\{-\tfrac12[n'(n'-1)]^{1/2}\delta_{n,n'-2}+(n+\tfrac12)\delta_{n,n'}+\tfrac12[n(n-1)]^{1/2}\delta_{n,n'+2}\}.
\end{aligned} \tag{4.6}$$

It follows that for the original time-dependent system

$$\begin{aligned}
\langle n|q^2|n'\rangle&=\nu^2\langle\bar n|Q^2|\bar n'\rangle\\
&=\nu^2\hbar\{\tfrac12[n'(n'-1)]^{1/2}\delta_{n,n'-2}+(n+\tfrac12)\delta_{n,n'}+\tfrac12[n(n-1)]^{1/2}\delta_{n,n'-2}\},\\
\langle n|p^2|n'\rangle&=\nu^{-2}\langle\bar n|P^2+M\nu\dot\nu(QP+PQ)+M^2\nu^2\dot\nu^2Q^2|\bar n'\rangle\\
&=(\hbar/\nu^2)\{\tfrac12(M^2\nu^2\dot\nu^2-2\mathrm iM\nu\dot\nu-1)[n'(n'-1)]^{1/2}\delta_{n,n'-2}\\
&\quad+(M^2\nu^2\dot\nu^2+1)(n+\tfrac12)\delta_{n,n'}\\
&\quad+\tfrac12(M^2\nu^2\dot\nu^2+2\mathrm iM\nu\dot\nu-1)[n(n-1)]^{1/2}\delta_{n,n'+2}\}.
\end{aligned} \tag{4.7}$$

The matrix elements of H are given by

$$\langle n|H|n'\rangle=\langle n|\tfrac12p^2/M+\tfrac12M\omega^2q^2|n'\rangle=\tfrac12M^{-1}\langle n|p^2|n'\rangle+\tfrac12M\omega^2\langle n|q^2|n'\rangle. \tag{4.8}$$

5. The examples of Colegrave and Abdalla

It is not a difficult task to recover the results of Colegrave and Abdalla by simply substituting their expressions for $M(t)$ in the appropriate formula. However, as will be seen below, there is some advantage in keeping the treatment general for the time being. Given an $M(t)$, our main task is to determine $\nu(t)$ which is defined in (3.9). To a lesser extent we are also interested in $\tau(t)$ which was also defined in (3.9), but, as it has not occurred in any of the expectation values or matrix elements, we shall concentrate on $\nu(t)$.

To obtain an explicit expression $\nu(t)$ we need to obtain explicit relationships from

$$t' = \int_{t_0}^{t} M^{-1}(s)\,\mathrm{d}s \tag{5.1}$$

where usually we shall set $t_0 = 0$ and

$$\ddot{\rho}(t') + N^2(t')\omega^2\rho(t') = \rho^{-3}(t'). \tag{5.2}$$

The solution of (5.2) is (cf Eliezer and Gray 1976)

$$\rho(t') = [A\sigma_1^2(t') + B\sigma_2^2(t') + 2C\sigma_1(t')\sigma_2(t')]^{1/2} \tag{5.3}$$

where $\sigma_1(t')$ and $\sigma_2(t')$ are two independent solutions of

$$\ddot{\sigma}(t') + N^2(t')\omega^2\sigma(t') = 0 \tag{5.4}$$

and the constants A, B and C are related according to

$$AB - C^2 = W^{-2}, \qquad W = \sigma_1(t')\dot{\sigma}_2(t') - \dot{\sigma}_1(t')\sigma_2(t'). \tag{5.5}$$

As far as $\nu(t)$ is concerned we may conflate the double exercise implied by (5.1) and (5.2) by means of the following transformations. Let

$$\sigma(t'(t)) = y(t). \tag{5.6}$$

Then (5.4) becomes

$$\ddot{y} + [\dot{M}(t)/M(t)]\dot{y} + \omega^2 y = 0 \tag{5.7}$$

where now a dot indicates differentiation with respect to t. From (5.3) we have

$$\nu(t) = [Ay_1^2(t) + By_2^2(t) + 2Cy_1(t)y_2(t)]^{1/2} \tag{5.8}$$

and the Wronskian in (5.5) is

$$W = M^{-1}(t)[y_1(t)\dot{y}_2(t) - \dot{y}_1(t)y_2(t)]. \tag{5.9}$$

A further change of variable can be made to reduce (5.7) to normal form. As $M(t)$ is non-negative on physical grounds, we may write

$$M(t) = \eta^2(t), \qquad \zeta(t) = \eta(t)y(t), \tag{5.10}$$

to transform (5.7) to

$$\ddot{\zeta} + (\omega^2 - \ddot{\eta}/\eta)\zeta = 0. \tag{5.11}$$

Whether we attempt to solve equation (5.7) or (5.11) is a matter of no great import. However, (5.11) does suggest a particularly simple set of selections of the variable mass. If $\ddot{\eta}$ is a constant multiple of η, both $M(t)$ and $\zeta(t)$ are easily determined. We have three possible cases:

(i) $\quad M(t) = \exp(2\alpha t), \qquad \ddot{\eta} = \alpha^2\eta,$

(ii) $\quad M(t) = (1+\alpha t)^2, \qquad \ddot{\eta} = 0,$

(iii) $\quad M(t) = \cos^2 \alpha t, \qquad \ddot{\eta} = -\alpha^2\eta.$

Cases (i) and (iii) have been dealt with by Colegrave and Abdalla (1981b, 1982). We will assume that for case (i), α is small enough for $\omega^2 - \alpha^2$ to be positive. From (5.11) we have

$$\zeta_1 = \sin \beta t, \qquad \zeta_2 = \cos \beta t, \tag{5.12}$$

where $\beta^2 = \omega^2 - \alpha^2$, ω^2, $\omega^2 + \alpha^2$ for cases (i), (ii) and (iii) respectively. In each case the Wronskian W has the value $-\beta$. So,

$$\nu(t) = [A\eta^{-2} \sin^2 \beta t + B\eta^{-2} \cos^2 \beta t + 2C\eta^{-2} \sin \beta t \cos \beta t]^{2}. \tag{5.13}$$

From (5.5) we see that

$$AB - C^2 = 1/\beta^2 \tag{5.14}$$

and, looking at (5.13), it is apparent that the choice

$$A = B = 1/\beta, \qquad C = 0, \tag{5.15}$$

gives the simple result

$$\nu(t) = \eta^{-1}(t)\beta^{-1/2}. \tag{5.16}$$

The expressions for $\langle n|V|n\rangle$ and $\langle n|T|n\rangle$ (V and T are the potential and kinetic energies respectively) in cases (i) and (iii) replicate those given by Colegrave and Abdalla. For case (ii) we have

$$\langle n|V|n\rangle = \tfrac{1}{2}M\omega^2\nu^2\hbar(n+\tfrac{1}{2}) = \tfrac{1}{2}\omega\hbar(n+\tfrac{1}{2}), \tag{5.17}$$

$$\begin{aligned}\langle n|T|n\rangle &= \tfrac{1}{2}M^{-1}(\nu^{-2} + M^2\dot{\nu}^2)\hbar(n+\tfrac{1}{2}) \\ &= \tfrac{1}{2}[\omega + (\alpha^2/\omega)(1+\alpha t)^{-2}]\hbar(n+\tfrac{1}{2}).\end{aligned} \tag{5.18}$$

6. Some other variable masses

The simplicity of the solution of (5.11) when $\ddot{\eta}/\eta$ is a constant does not persist when $\ddot{\eta}/\eta$ is non-constant. However, with a little ingenuity and the assistance of a reference work such as Kamke (1959), it is possible to obtain a few solutions for various varying masses with not much difficulty. As examples we have the following.

(i)
$$\eta(t) = \omega \cosh \omega t - t^{-1} \sinh \omega t,$$
$$\zeta_1(t) = t^2, \qquad \zeta_2(t) = t^{-1},$$
$$W = -3,$$
$$\nu^2(t) = (\omega \cosh \omega t - t^{-1} \sinh \omega t)(t^6 + 1)/3t^2.$$

(ii)
$$\eta(t) = t^2,$$
$$\zeta_1(t) = \omega^2 \sin \omega t + \omega t^{-1} \cos \omega t, \qquad \zeta_2(t) = \omega^2 \cos \omega t - \omega t^{-1} \sin \omega t,$$
$$W = -\omega^3,$$
$$\nu(t) = (\omega^3 t^2)^{-1/2}(\omega^2 + t^{-2})^{1/2}.$$

More generally, for

$$\eta(t) = t^n, \qquad \zeta(t) = t^m(t^{-1}D)^m(c_1 \sin \omega t + c_2 \cos \omega t),$$

where D represents $\mathrm{d}/\mathrm{d}t$ and

$$m=\begin{cases} n, & n\in Z^+\cup\{0\},\\ -n+1, & n\in Z^-, \end{cases}$$

(cf Kamke 1959, p 435, 2.153).

(iii) $$\eta(t)=\cos^n \alpha t,$$

$$\zeta(t)=\cos^n \alpha t(\cos^{-1}\alpha t D)^n(C_1\sin\beta t+C_2\cos\beta t),$$

where $D=\mathrm{d}/\mathrm{d}(\alpha t)$ and $\beta^2=\omega^2+\alpha^2n^2$. The case $n=1$ was given in § 5. For $n=2$,

$$\nu(t)=\beta^{-1/2}\cos^{-2}\alpha t(1+\beta^{-2}\tan^2\alpha t)^{1/2}.$$

(See Kamke 1959, p 504, 4.20.)

(iv) $$\eta(t)=\exp(-\tfrac{1}{2}\alpha^2t^2),\qquad \zeta(t)=\exp(-\tfrac{1}{2}\alpha^2t^2)(y_1(\alpha t)+y_2(\alpha t))$$

where

$$y_1(x)=1+\sum_{n=1}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!}\omega^2(\omega^2-4)\ldots(\omega^2-4n+4),$$

$$y_2(x)=x+\sum_{n=1}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}(\omega^2-2)(\omega^2-6)\ldots(\omega^2-4n+2).$$

(v) $$\eta(t)=H_n(x)\exp(-\tfrac{1}{2}t^2).$$

The solution has the same form as in (iv) with ω^2 replaced by $\beta^2=\omega^2+(2n+1)$.

(vi) $$\eta(t)=T_n(\cos t)$$

($T_n(\cos t)$ is a Chebyshev polynomial of the first kind).

$$\zeta_1=\sin\beta t,\qquad \zeta_2=\cos\beta t,\qquad \beta^2=\omega^2+n^2,$$

$$\nu(t)=\beta^{-1/2}T_n^{-1}(\cos t).$$

(vii) $$\eta(t)=(\alpha t)^{1/2}J_\nu(\alpha t),$$

$$\zeta_1(t)=(\alpha t)^{1/2}J_\beta(\alpha t),\qquad \zeta_2(t)=(\alpha t)^{1/2}Y_\beta(\alpha t),\qquad \beta^2=\nu^2-\omega^2,$$

$$\nu(t)=(\pi/2\alpha)^{1/2}J_\nu^{-1}(\alpha t)M_\beta(\alpha t),$$

where

$$M_\beta^2(\alpha t)=J_\beta^2(\alpha t)+Y_\beta^2(\alpha t).$$

7. Discussion

The examples given in §§ 5 and 6 do have some objectionable features from a physical viewpoint. It will be recalled that the function $\eta(t)$ is the square root of the mass of the oscillator. In the various examples, the mass can become infinite or zero at some time which may be finite. Colegrave and Abdalla (1982) have remarked that a mass which becomes zero could arise in an ideal situation for a Fabry–Pérot cavity. However, they do state in their conclusion that a gentler variation of mass which avoids such extremes is to be preferred. The effect of a zero in $M(t)$ is to change the time scale in (q', p', t') space. This is easily illustrated with the function used by Colegrave and Abdalla (1982). There $M(t) = \cos^2 \alpha t$ and so

$$t' = \int^t \sec^2 \alpha s \, ds = \alpha^{-1} \tan \alpha t,$$

i.e. the time between successive zeros of $M(t)$, π/α, is expanded to infinity in terms of t'.

The method adopted in this note does provide a procedure for dealing with the case of the mass varying periodically without zeros. For reasons which will become apparent shortly we merely outline the procedure. A suitably periodic non-zero mass can be given by writing

$$\eta(t) = a + b \cos \alpha t, \qquad a > |b| > 0. \tag{7.1}$$

Equation (5.11) is then

$$\zeta'' + [\omega^2 + \alpha^2 - a\alpha^2/(a + b \cos \alpha t)]\zeta = 0. \tag{7.2}$$

The change of variable

$$(t, \zeta) \to (x, y : x = \tfrac{1}{2}\alpha t, y(x) = \zeta(t)) \tag{7.3}$$

transforms (7.2) to

$$y'' + 4[\beta^2 - 1/(1 + \sigma \cos 2x)]y = 0 \tag{7.4}$$

where $\beta^2 = (\omega^2 + \alpha^2)/\alpha^2$ and $|\sigma| = |b/a| < 1$.

Equation (7.4) is of the form of Hill's equation for which a standard treatment exists (cf Ince 1956, 383ff). However, that treatment is beyond the bounds of a note. Alternatively, equation (7.2) may be transformed to Heun's equation (cf Kamke 1959, 2.329, p 485), the treatment of which is also rather lengthy. We merely wish to make the point that a physically realistic periodically varying mass can be treated in terms of known differential equations.

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